DUALITY THEOREMS FOR SOME COMMUTATIVE SEMIGROUPS(1)

BY

C. W. AUSTIN(2)

1. **Introduction.** The Pontrjagin duality theorem asserts that a locally compact abelian group G can be identified in a natural way with its second dual $G^{\wedge\wedge}$; specifically, every character of the dual group $G^{\wedge}$ has the form $f(\chi) = \chi(x)$ for all $\chi \in G^{\wedge}$, where x is some element of G, and if the function f defined above is denoted by $\tilde{x}$, the natural mapping $x \to \tilde{x}$ is an isomorphism and a homeomorphism of G onto $G^{\wedge\wedge}$. In their paper [3], E. Hewitt and H. S. Zuckerman raise the question of whether an analogue of this theorem exists for (discrete) commutative semigroups. For a semigroup G, the dual object is the set $G^{\wedge}$ of all semicharacters of G (as defined below): the set $G^{\wedge}$ is a semigroup if G has an identity and in certain other cases. The principal result is that, for discrete abelian G, we have $G^{\wedge\wedge} = G$ if and only if G has identity and is a union of groups. A similar but less complete result is obtained for commutative compact Hausdorff G.

1.1 Definition. A commutative semigroup is a nonempty set G together with a mapping $(x, y) \to xy$ on $G \times G$ to G such that $(xy)z = x(yz)$ and $xy = yx$ whenever $x, y, z \in G$. If G is also a topological space and the mapping $(x, y) \to xy$ is continuous on $G \times G$, then G is a commutative topological semigroup.

1.2 Definition. A semicharacter of a commutative topological semigroup G is a bounded, continuous, complex-valued function χ on G such that $\chi(x) \neq 0$ for some x in G and $\chi(xy) = \chi(x)\chi(y)$ for all x, y in G. The set of all semicharacters of G is denoted by $G^{\wedge}$.

1.3 Definition. Let G be a commutative topological semigroup. Let C be compact subset of G and let $\varepsilon > 0$. For $\chi \in G^{\wedge}$ we define

$$U_{C,\varepsilon}(\chi) = \{\psi \in G^{\wedge} : |\psi(x) - \chi(x)| < \varepsilon \text{ for all } x \in C\}$$

and we define the topology of $G^{\wedge}$ to be the smallest which makes all the sets $U_{C,\varepsilon}(\chi)$ open.

Presented to the Society, June 6, 1961 under the title *Duality theorems for semigroups;* received by the editors May 8, 1962.

(1) The results presented here appeared in the author's doctoral dissertation, University of Washington, June, 1962, the research for which was supported in part by National Science Foundation Grant NSF-G10738.

(2) The author is indebted to Professors Edwin Hewitt and Herbert S. Zuckerman for their support and encouragement during the preparation of this paper.

1.4 REMARK. The pointwise product of two semicharacters of G is either a semicharacter of G or is identically zero. If the latter never occurs, then $G^\wedge$ is a commutative Hausdorff semigroup. In particular, if G has identity e, then $\chi(e) = 1$ for all $\chi \in G^\wedge$.

1.5 DEFINITION. An ideal of the commutative semigroup G is a (possibly void) subset I of G such that $GI \subset I$. A prime ideal is an ideal J such that $G \setminus J$ is a semigroup.

1.6 DEFINITION. Let $\chi \in G^\wedge$ and define the null set and the interior set of χ to be, respectively, $N(\chi) = \{x \in G : \chi(x) = 0\}$ and $I(\chi) = \{x \in G : |\chi(x)| < 1\}$.

1.7 REMARK. If $\chi \in G^\wedge$, then $N(\chi)$ is a closed prime ideal and $I(\chi)$ is an open prime ideal of G.

1.8 NOTATION. Let $G^* = \{\chi \in G^\wedge : |\chi(x)| = 0 \text{ or } 1 \text{ for all } x \in G\}$. (Equivalently, $G^* = \{\chi \in G^\wedge : N(\chi) = I(\chi)\}$.)

1.9 NOTATION. If G and H are topological semigroups, then the notation $G \cong H$ will mean that there is an algebraic isomorphism of G onto H which is also a homeomorphism.

1.10 REMARK. If e and f are idempotent elements of a commutative semigroup G, we write $e \leqq f$ if and only if $ef = e$. The set of idempotents of G is partially ordered by this relation.

2. **Commutative compact Hausdorff semigroups.** There is an elegant structure theory, due largely to Š. Schwarz [8; 9; 10], for commutative compact Hausdorff semigroups. Some of these results are stated below without proof. Throughout this section G denotes a commutative compact Hausdorff semigroup.

2.1 THEOREM. *For each $x \in G$ the closure of the set $\{x, x^2, x^3, \cdots\}$ contains a unique idempotent.*

2.2 DEFINITION. If $x \in G$ and e is the unique idempotent in $\{x, x^2, x^3, \cdots\}^-$, then x is said to belong to e.

2.3 THEOREM. *Let $E = \{e_\alpha : \alpha \in A\}$ be the set of idempotents of G, and for each index $\alpha \in A$ let P_α be the set of elements of G which belong to e_α. If $e_\gamma = e_\alpha e_\beta$ then $P_\alpha P_\beta \subset P_\gamma$. In particular, P_α is a semigroup.*

2.4 THEOREM. *To each idempotent e_α of G there corresponds a unique maximal subgroup of G which contains e_α, namely $G_\alpha = \{x \in G;\ x \in P_\alpha$ and $e_\alpha x = x\}$. The group G_α is closed but the semigroup P_α need not be.*

2.5 REMARK. The semigroups P_α are disjoint, and $G = \bigcup_{\alpha \in A} P_\alpha$. This decomposition of G will be referred to as the Schwarz decomposition, and the semigroup P_α will be called the Schwarz semigroup belonging to e_α. The groups G_α will be called the maximal groups of G. Since these are unique, it follows that G is the union of disjoint groups if and only if $P_\alpha = G_\alpha$ for all $\alpha \in A$.

2.6 THEOREM. *The semigroup G has a least idempotent in the sense of* 1.10.

2.7 THEOREM. *There is a one-to-one correspondence between idempotents of G and open prime ideals of G given by $e_\alpha \to J_\alpha$, where*

2.7.1 $$J_\alpha = \bigcup \{P_\beta : e_\alpha e_\beta \neq e_\alpha\}.$$

2.8 DEFINITION. The prime ideal J_α, whose complement is the union of all Schwarz semigroups belonging to idempotents $e_\beta \geqq e_\alpha$, is called the associated open prime ideal of e_α. The idempotent e_α is called a generating idempotent if and only if its associated open prime ideal is closed.

3. **Some properties of** $G^\wedge$. In this section G denotes a commutative topological semigroup. Certain theorems about the topological or algebraic structure of $G^\wedge$, which are needed in the sequel, are proved.

3.1 THEOREM. *If G is discrete and has an identity, then $G^\wedge$ is a commutative compact Hausdorff semigroup.*

Proof. Compact subsets of G are finite, so that the topology of $G^\wedge$ is its relative topology as a subspace of D^G with its usual product topology, where D is the set of complex numbers z with $|z| \leqq 1$. Now D^G is compact by Tychonoff's theorem, and $G^\wedge$ is easily seen to be a closed subset of D^G. (The existence of an identity insures that 0 is not a limit point of $G^\wedge$.) The remainder of the proof follows from 1.4.

3.2 THEOREM. *If J is an open and closed prime ideal of G then ε_J, the characteristic function of $G \setminus J$, is an idempotent of $G^\wedge$, and every idempotent of $G^\wedge$ has this form. In particular, if G is discrete then $\{\varepsilon_J : J$ is a prime ideal of $G\}$ is the set of all idempotents of $G^\wedge$.*

Proof. Obvious.

3.3 THEOREM. *Let G be such that $G^\wedge$ is a commutative compact Hausdorff semigroup. Let $\chi \in G^\wedge$. Then χ belongs to the idempotent ε_J if and only if $I(\chi) = J$.*

Proof. By 2.1 and 2.2, χ belongs to a unique idempotent ε_J, and it suffices to prove that $J = I(\chi)$.

If $x \in I(\chi)$, then $1 - |\chi(x)| = \delta > 0$. Let $U = \{\psi \in G^\wedge : |\psi(x) - \varepsilon_J(x)| < \delta\}$. Then U is a neighborhood of ε_J, so that $\chi^n \in U$ for some positive integer n. Thus $\varepsilon_J(x) < |X(x)|^n + \delta \leqq 1$; it follows that $\varepsilon_J(x) = 0$ and that $x \in J$.

Conversely, let $y \in J$ and let $V = \{\psi \in G^\wedge : |\psi(y) - \varepsilon_J(y)| < 1\}$. Then $\chi^m \in V$ for some m, and $|\chi(y)|^m = |\chi^m(y) - \varepsilon_J(y)| < 1$. Thus $y \in I(\chi)$.

3.4 THEOREM. *If G is compact, then G^* is a discrete subspace of $G^\wedge$.*

Proof. Let $\chi \in G^*$. We will show that $U_{G,1/2}(\chi) \cap G^* = \{\chi\}$. If $\psi \in G^*$ and $\psi \neq \chi$, then $\psi(x) \neq \chi(x)$ for some $x \in G$. If $\psi(x) = 0$ and $|\chi(x)| = 1$ or *vice versa*, then $\psi \notin U_{G,1/2}(\chi)$. Thus we may suppose that $\psi(x) = e^{ia}$ and $\chi(x) = e^{ib}$, where $0 \leqq a < b < 2\pi$. Let $c = b - a$. Then there is a positive integer n such that either $\pi/4 \leqq nc < 7\pi/4$ or $\pi/4 \leqq 2n\pi - nc < 7\pi/4$. In either case, $|\psi(x^n) - \chi(x^n)| = |e^{ina} - e^{inb}| = |1 - e^{inc}| = |e^{i(2n\pi - nc)} - 1| > 1/2$.

4. **Inverse semigroups.** Let G be a semigroup which is the union of groups, i.e. every element of G is contained in a subgroup of G. Then G is actually the union of pairwise disjoint groups. In fact, let E be the set of idempotents of G and for each $e \in E$ let $G_e = \{x \in G : ex = xe = x$ and there exists $x' \in G$ such that $xx' = x'x = e\}$. Then the sets G_e are pairwise disjoint groups and $G = \bigcup_{e \in E} G_e$. Furthermore, if G is commutative we have $G_e G_f \subset G_{ef}$. In this case G may be called a semilattice of groups (cf. Clifford [1]). Other concepts which have been studied and which are, in the presence of commutativity, equivalent to the property described above are inverse semigroup [5; 7; 11] and semigroup admitting relative inverses [2]. We shall use the term "inverse semigroup" to describe a commutative semigroup which is is the union of groups. In the remainder of this section G will denote a commutative inverse (topological) semigroup, E will denote its set of idempotents, and for each $e \in E$ we define G_e as above.

4.1 Lemma. *For each idempotent e, the set*

4.1.1 $$J_e = \bigcup\{G_f : f \in E \text{ and } ef \neq e\}$$

is a prime ideal of G.

Proof. It is easy to see that $efg = e$ if and only if $ef = e$ and $eg = e$, where $f, g \in E$. From this it follows that $xy \in G \setminus J_e$ if and only if $x \in G \setminus J_e$ and $y \in G \setminus J_e$. This in turn implies that J_e is a prime ideal.

4.2 Remark. The groups G_e are topological semigroups (with the relative topology of G) which happen to be groups. We may define the set of semicharacters $G_e^\wedge$ as in 1.2, with its topology as in 1.3. In this case $G_e^\wedge$ is a topological group and, if G_e is a topological group (i.e., inversion is continuous) then $G_e^\wedge$ is its character group. In the cases we shall study, G_e is either compact or discrete. In each of these cases, G_e is a topological group. (For the compact case, see Numakura [6].)

4.3 Theorem. *Suppose that the prime ideal J_e is open and closed. Then every semicharacter χ_e of G_e admits a unique extension to a semicharacter χ of G for which $N(\chi) = J_e$, namely*

4.2.1 $$\chi(x) = \begin{cases} 0 \text{ if } x \in J_e, \\ \chi_e(ex) \text{ if } x \in G \setminus J_e. \end{cases}$$

Furthermore, if $\mathfrak{S}_{J_e} = \{\chi \in G^\wedge : N(\chi) = J_e\}$, then $\mathfrak{S}_{J_e} \cong G_e^\wedge$ under the mapping $\chi \to \chi_e = \chi | G_e$.

Proof. Since $ex \in G_e$ whenever $x \in G \setminus J_e$, χ is well defined. The details of the proof that χ is bounded, multiplicative and not identically zero are omitted (cf. [3, Theorem 5.5]). Obviously $N(\chi) = J_e$, and χ is continuous because the mapping $x \to ex$ is continuous on $G \setminus J_e$, because χ_e is continuous on G_e and because J_e is open and closed. The uniqueness of χ is easily proved.

Since the mapping $\chi \to \chi \mid G_e$ is obviously a homomorphism of $\mathfrak{S}_{J_e}$ into G_e, and since the above argument shows that it is one-to-one and onto, it remains only to show that this mapping is a homeomorphism. Given $\varepsilon > 0$ and C a compact subset of G_e, the open set $U_{C,\varepsilon}(\chi)$ is mapped into $U_{C,\varepsilon}(\chi_e)$. To show continuity of the inverse mapping, let $\varepsilon > 0$ and a compact subset C of G be given. Let $C' = [C \cap (G \setminus J_e)]e$; then C' is a compact subset of G_e and $U_{C',\varepsilon}(\chi_e)$ is mapped into $U_{C,\varepsilon}(\chi)$.

4.4 Remark. If an ideal I of G contains any element of G_e, then we must have $G_e \subset I$. Thus every ideal of G is itself the union of maximal groups G_e, and in fact we may write

4.4.1 $$I = \bigcup \{G_e : e \in E \cap I\}.$$

5. **Duality theorem for discrete commutative semigroups.** Throughout this section, unless otherwise specified, G will denote a discrete commutative inverse semigroup with identity. As in §4, E will denote the set of idempotents of G, and G_e will denote the maximal group containing e. We will prove that $G \cong G^{\wedge\wedge}$ by the natural mapping $x \to \tilde{x}$, where $\tilde{x}(\chi) = \chi(x)$. It is obvious that each function $\tilde{x}$ is a semicharacter of $G^{\wedge}$ and that the mapping $x \to \tilde{x}$ is a homomorphism of G into $G^{\wedge\wedge}$. Since G is the union of disjoint groups, it follows that for all $x, y \in G$, $x^2 = xy = y^2$ implies $x = y$. This condition is necessary and sufficient for $G^{\wedge}$ to separate points of G [3, Theorems 3.5 and 5.8], and thus the mapping $x \to \tilde{x}$ is one-to-one. It will be seen below that $G^{\wedge\wedge}$ is discrete, so that all that remains to be shown is that every semicharacter of $G^{\wedge}$ has the form $\tilde{x}$ for some $x \in G$.

By Theorem 3.1, $G^{\wedge}$ is a commutative compact Hausdorff semigroup. The idempotents of $G^{\wedge}$ are the functions ε_J defined in Theorem 3.2. The following lemma describes the Schwarz decomposition of $G^{\wedge}$.

5.1 Lemma. *For each prime ideal J of G, let*

5.1.1 $$\mathfrak{S}_J = \{\chi \in G^{\wedge} : N(\chi) = J\}.$$

Then $\mathfrak{S}_J$ is the semigroup of elements of $G^{\wedge}$ which belongs to the idempotent ε_J. Furthermore, each semigroup $\mathfrak{S}_J$ is a closed subgroup of $G^{\wedge}$.

Proof. Since G is the union of groups G_e and since a semicharacter of G is, when restricted to G_e, either a character of G_e or zero, it follows that $N(\chi) = I(\chi)$ for all $\chi \in G^{\wedge}$. The first assertion now follows at once from Theorem 3.3. For

$\chi \in \mathfrak{S}_J$, we have $\chi\bar{\chi} = \varepsilon_J$ and $\varepsilon_J \chi = \chi$, so that $\mathfrak{S}_J$ is a group. If $\chi \in G^\wedge \setminus \mathfrak{S}_J$ we may choose $x \in N(\chi) \setminus J$ (or $J \setminus N(\chi)$). Then $U_{x,1}(\chi) \subset G^\wedge \setminus \mathfrak{S}_J$. (See also Theorem 2.4.)

5.2 Lemma. *$G^{\wedge\wedge}$ is discrete.*

Proof. Since $G^\wedge$ is the union of groups, it follows that $G^{\wedge\wedge} = G^{\wedge *}$. By Theorem 3.4, $G^{\wedge\wedge}$ is discrete.

5.3 Lemma. *Every open prime ideal of $G^\wedge$ has the form*

5.3.1 $$\mathfrak{R}_J = \bigcup \{\mathfrak{S}_I : I \text{ is a prime ideal of } G, I \not\subset J\}$$

for some prime ideal J of G, and every such set $\mathfrak{R}_J$ is an open prime ideal of $G^\wedge$.

Proof. Since $\varepsilon_I \varepsilon_J = \varepsilon_J$ if and only if $I \subset J$, this is merely a special case of Theorem 2.7.

5.4 Remark. If $\chi \in G^\wedge$ and $N(\chi) = I$, then $\chi \in \mathfrak{R}_J$ if and only if $I \not\subset J$. We thus have the more useful characterization:

5.4.1 $$\mathfrak{R}_J = \{\chi \in G^\wedge : N(\chi) \not\subset J\}.$$

5.5 Definition. For each prime ideal J of G, let

5.5.1 $$J^0 = \bigcup \{I : I \text{ is a prime ideal of } G,\ I \not\subset J\}.$$

5.6 Remark. Since $J^0 = \bigcup \{N(\chi) : \chi \in \mathfrak{R}_J\}$, we may also write:

5.6.1 $$J^0 = \{x \in G : \chi(x) = 0 \quad \text{for all } \chi \in \mathfrak{R}_J\}.$$

5.7 Example. Let G be the set of natural numbers with the operation $mn = \max(m,n)$. The prime ideals of G are the empty set and the sets $J_n = \{n+1, n+2, \cdots\}$ for $n = 1, 2, \cdots$. If $J = \emptyset$, then $J^0 = \bigcap_{n=1}^{\infty} J_n = \emptyset$. This example shows that J^0 need not be nonvoid; however, it will turn out that J^0 is nonvoid in the case which interests us.

Since $G^{\wedge\wedge} = G^{\wedge *}$, $N(\phi)$ is an open and closed prime ideal of G whenever $\phi \in G^{\wedge\wedge}$. In particular, there is a one-to-one correspondence between idempotents of $G^{\wedge\wedge}$ and open and closed prime ideals of $G^\wedge$. The following lemma gives a criterion for the open prime ideal $\mathfrak{R}_J$ to be closed.

5.8 Lemma. *$\mathfrak{R}_J$ is closed if and only if $J^0 \setminus J \neq \emptyset$.*

Proof. Let $x \in J^0 \setminus J$. If $\chi(x) = 0$ then $N(\chi) \not\subset J$ and, by 5.4, we have $\chi \in \mathfrak{R}_J$. By 5.6, $\chi \in \mathfrak{R}_J$ implies that $\chi(x) = 0$. Thus $\mathfrak{R}_J = \tilde{x}^{-1}(0)$, which is closed because $\tilde{x}$ is continuous.

Now suppose $J^0 \setminus J = \emptyset$. Then for each $x \in G \setminus J$ there exists $\chi \in \mathfrak{R}_J$ such that $\chi(x) \neq 0$. Let F be a finite subset of G and let $\delta > 0$. We show that

$U_{F,\delta}(\varepsilon_J) \cap \mathfrak{R}_J \neq \emptyset$, and hence that $\mathfrak{R}_J$ is not closed. Let $F \setminus J = \{x_1, \cdots, x_n\}$ and let $F \cap J = \{x_{n+1}, \cdots, x_m\}$. Since the product $x_1 \cdots x_n$ is in $G \setminus J$, there exists $\chi \in \mathfrak{R}_J$ such that $0 \neq \chi(x_1 \cdots x_n) = \chi(x_1) \cdots \chi(x_n)$. Let $\psi = \varepsilon_J \chi \bar{\chi}$. Then $\psi \in \mathfrak{R}_J$ and $\psi(x_i) = \varepsilon_J(x_i)$ for $i = 1, \cdots, m$.

5.9 Lemma. *$\mathfrak{R}_J$ is closed if and only if $J^0 \setminus J = G_e$ for some $e \in E$.*

Proof. Sufficiency is obvious. Now let $\mathfrak{R}_J$ be closed. Then $J^0 \setminus J \neq \emptyset$. Since J^0 and J are ideals, each of them must be a union of maximal groups. Hence so is $J^0 \setminus J$. If e and f are idempotents in $J^0 \setminus J$, then $\chi(e) = 0 = \chi(f)$ for $\chi \in \mathfrak{R}_J$ while $\chi(e) = 1 = \chi(f)$ for $\chi \in G^\wedge \setminus \mathfrak{R}_J$. Since $G^\wedge$ separates points it follows that $f = e$ and that $J^0 \setminus J = G_e$.

5.10 Lemma. *Let $e \in E$ and let J_e be as in 4.1.1. Then $J^0 \setminus J = G_e$ if and only if $J = J_e$.*

Proof. Suppose $J^0 \setminus J = G_e$. If f is an idempotent and $f \in J$ then $ef \in J$ also, and $ef \neq e$. But if $f \in G \setminus J$, then $ef \in J^0 \setminus J$ because J^0 is an ideal and $G \setminus J$ is a semigroup. Since e is the unique idempotent of $J^0 \setminus J$, it follows that $ef = e$. Thus $f \in J$ if and only if $ef \neq e$; that is, $J = J_e$.

Conversely, suppose $J = J_e$. If I is a prime ideal and $I \not\subset J_e$, then there is an idempotent $f \in I$ such that $ef = e$; this in turn implies that $e \in I$. Thus $e \in J_e^0 \setminus J_e$; by Lemma 5.9 $J_e^0 \setminus J_e = G_e$.

5.11 Lemma. *The idempotent ε_J of $G^\wedge$ is a generating idempotent if and only if $N(\varepsilon_J) = J_e$ for some $e \in E$.*

Proof. Since $J = N(\varepsilon_J)$ and $\mathfrak{R}_J$ is the associated open prime ideal of ε_J, the lemma is an immediate consequence of 2.8, 5.9 and 5.10.

5.12 Lemma. *For each $e \in E$, $\mathfrak{R}_{J_e} = N(\tilde{e})$.*

Proof. Since $e \in J_e^0 \setminus J_e$, it follows from 5.4 and 5.6 that $\chi \in \mathfrak{R}_{J_e}$ if and only if $\tilde{e}(\chi) = \chi(e) = 0$.

5.13 Remark. Lemmas 5.8, 5.9 and 5.10 identify the open and closed prime ideals of $G^\wedge$ as precisely the ideals $\mathfrak{R}_{J_e}$ for $e \in E$. (Lemma 5.11 is superfluous for this purpose, but we shall need it in §6.) Lemma 5.12 implies that every idempotent of $G^{\wedge\wedge}$ has the form $\tilde{e}$ for some $e \in E$. Let us now define

5.13.1 $$P_e = \{\phi \in G^{\wedge\wedge} : N(\phi) = \mathfrak{R}_{J_e}\}$$

and also

5.13.2 $$\tilde{G}_e = \{\tilde{x} \in G^{\wedge\wedge} : x \in G_e\}.$$

It is clear that each set P_e is a group, that these groups are disjoint and that $G^{\wedge\wedge} = \bigcup_{e \in E} P_e$. It now suffices to show that $P_e = \tilde{G}_e$ for each $e \in E$.

5.14 Lemma. *For each $e \in E$ we have $P_e = \tilde{G}_e$.*

Proof. Since for $x \in G_e$ and $\chi \in G^{\wedge}$ we have $\chi(x) = 0$ if and only if $\chi(e) = 0$, it follows that $N(\tilde{x}) = N(\tilde{e}) = \mathfrak{R}_{J_e}$. Thus $\tilde{G}_e \subset P_e$.

Now let $\phi \in P_e$. Then $\phi \mid \mathfrak{S}_{J_e}$ is a character of the compact group $\mathfrak{S}_{J_e}$. By Theorem 4.3 we have $\mathfrak{S}_{J_e} \cong G_e^{\wedge}$ by the mapping $\chi \to \chi_e = \chi \mid G_e$. Thus the function $\chi_e \to \phi(\chi)$ is a character of $G_e^{\wedge}$. By the Pontrjagin duality theorem there exists $x \in G_e$ such that $\phi(\chi) = \chi_e(x) = \chi(x)$ for $\chi \in \mathfrak{S}_{J_e}$. Now $\phi \mid \mathfrak{S}_{J_e} = \tilde{x} \mid \mathfrak{S}_{J_e}$, and the first assertion of Theorem 4.3, applied to the inverse semigroup $G^{\wedge}$, implies that $\phi = \tilde{x}$. (Here $\mathfrak{R}_{J_e}$ takes the place of J_e, and $\mathfrak{S}_{J_e}$ takes the place of G_e.)

Before stating the main result of this section, we give the following near converse to the results so far obtained.

5.15 Lemma. *Let G be any discrete commutative semigroup such that $G^{\wedge}$ is a semigroup. If the natural mapping $x \to \tilde{x}$ is an algebraic isomorphism of G onto $G^{\wedge\wedge}$ then G is an inverse semigroup with identity.*

Proof. Since $G^{\wedge\wedge}$ has an identity, so does G. The fact that the mapping $x \to \tilde{x}$ is one-to-one implies that $G^{\wedge}$ separates points of G. By [3, Theorem 5.6], G^* also separates points of G.

For $x \in G$, let x' be the element whose image is the complex conjugate of $\tilde{x}$ in $G^{\wedge\wedge}$, and let $e = xx'$. If $\chi \in G^*$, then $\chi(e) = |\chi(x)|^2$; thus $\chi(e) = 1$ if $\chi(x) \neq 0$ and $\chi(e) = 0$ if $\chi(x) = 0$. Then $\chi(ex) = \chi(e)\chi(x) = \chi(x)$, whatever the value of $\chi(x)$. Since G^* separates points, it follows that $ex = x$. The existence of e and x' for an arbitrary element $x \in G$, such that $ex = x$ and $xx' = e$, implies that G is an inverse semigroup.

Combining the results of 5.1 through 5.13 with the above lemma, we have the following duality theorem for discrete commutative semigroups.

5.16 Theorem. *Let G be a discrete commutative inverse semigroup with identity. Then $G \cong G^{\wedge\wedge}$ under the natural mapping $x \to \tilde{x}$. If G is any discrete commutative semigroup such that $G^{\wedge}$ is a semigroup, and if the mapping $x \to \tilde{x}$ is an isomorphism of G onto $G^{\wedge\wedge}$, then G is an inverse semigroup with identity.*

This result was obtained by Hewitt and Zuckerman for finite commutative semigroups [4, Theorem 3.17].

5.17 Remark. In [11], R. J. Warne and L. K. Williams prove a duality theorem for idempotent "characters" of a commutative inverse semigroup G with identity, where a *character* of G is defined to be a multiplicative complex-valued function on G (and not identically zero if G has an identity). No topology is introduced in $G^{\wedge}$, the set of characters of G. Their theorem (1.13 of [11]) includes the hypothesis that every nonvoid subset of the semilattice of idempotents of G has a maximal element and a minimal element. The existence of the minimal element implies that every open prime ideal of $G^\wedge$ is closed (since every prime ideal of G has the form J_e; see 5.13). The existence of the maximal element can be shown to imply that every prime ideal of $G^\wedge$ is open. Thus the hypothesis of Warne and Williams implies that every idempotent character (in the sense of [11]) of $G^\wedge$ is continuous on $G^\wedge$ and is thus idempotent of $G^{\wedge\wedge}$. Corollary 1.13 of [11] now follows from 5.12.

6. **Duality theorem for commutative compact Hausdorff semigroups.** In this section we prove a theorem similar to, but weaker than, Theorem 5.16. In the case of a discrete commutative semigroup G, we made use of two results which are not available in the present case: (1) if $G^\wedge$ separates points of G then G^* also separates points; (2) if G is an inverse semigroup then $G^\wedge$ separates points.

The second statement is not necessarily true for a commutative compact Hausdorff semigroup G, as is shown by the example $G = [0,1]$ with $xy = \max(x, y)$ and the usual topology. This semigroup has only one semicharacter. The status of the first result is not known to us. Lacking these results, we are forced to introduce the hypothesis that $G^\wedge$ separates points, and we are unable to prove an analogue of 5.15.

Throughout this section, G will denote a commutative compact Hausdorff semigroup with identity, which is the union of groups and is such that $G^\wedge$ separates points. We will prove that $G \cong G^{\wedge\wedge}$ under the natural mapping $x \to \tilde{x}$.

We adopt the notation of §2. Since G is the union of groups we may write $G = \bigcup_{\alpha \in A} G_\alpha$, where G_α is the unique maximal group containing e_α. By Theorem 2.4, each group G_α is closed, hence is a compact topological group.

We first describe the structure of $G^\wedge$.

6.1 Lemma. *$G^\wedge$ is a discrete commutative inverse semigroup with identity. There is a one-to-one correpondence between generating idempotents of G and idempotents of $G^\wedge$ given by $e_\alpha \to \varepsilon_\alpha$, where*

6.1.1 $$\varepsilon_\alpha(x) = \begin{cases} 1 & \text{if } x \in G \setminus J_\alpha, \\ 0 & \text{if } x \in J_\alpha. \end{cases}$$

The set

6.1.2 $$\mathfrak{G}_\alpha = \{\chi \in G : N(\chi) = J_\alpha\}$$

is the maximal group containing ε_α.

Proof. The existence of an identity in G insures that $G^\wedge$ is a semigroup. Since G is an inverse semigroup we have $G^\wedge = G^*$ and thus $G^\wedge$ is discrete, by 3.4. The rest of the lemma is obvious.

6.2 Remark. By Theorem 3.1 we know that $G^{\wedge\wedge}$ is also a commutative compact Hausdorff semigroup; since $G^\wedge$ is an inverse semigroup, so is $G^{\wedge\wedge}$.

The natural mapping $x \to \tilde{x}$ is an isomorphism of G into $G^{\wedge\wedge}$; we must show that it is bi-continuous and onto.

6.3 Lemma. *The weak topology induced in G by $G^{\wedge}$ coincides with the given compact topology.*

Proof. The weak topology is a Hausdorff topology (because $G^{\wedge}$ separates points) which is smaller than the compact topology. Hence these topologies are the same.

6.4 Lemma. *Let $\tilde{G}$ denote the image of G under the natural mapping $x \to \tilde{x}$. Then $G \cong \tilde{G}$ under this mapping.*

Proof. Since compact subsets of $G^{\wedge}$ are finite, the image, under the natural mapping, of the weak topology induced by $G^{\wedge}$ is the topology of $\tilde{G}$ as a subspace of $G^{\wedge\wedge}$.

6.5 Lemma. *Let e_α be any idempotent of G, and let $\chi \in G^{\wedge}$. Then $\chi(e_\alpha) = 1$ if and only if $N(\chi) \subset J_\alpha$.*

Proof. If $\chi(e_\alpha) = 1$ and $e_\alpha \leqq e_\beta$, then $\chi(e_\beta) = \chi(e_\beta)\chi(e_\alpha) = \chi(e_\beta e_\alpha) = \chi(e_\alpha) = 1$. This implies that $G_\beta \subset G \setminus N(\chi)$. Now $G \setminus J_\alpha = \bigcup \{G_\beta : e_\alpha \leqq e_\beta\}$, so $G \setminus J_\alpha \subset G \setminus N(\chi)$. The converse is obvious, since $e_\alpha \in G \setminus J_\alpha$.

6.6 Lemma. *Every generating idempotent of $G^{\wedge\wedge}$ has the form $\tilde{e}_\alpha$ for some generating idempotent $e_\alpha \in G$.*

Proof. The discrete commutative inverse semigroup $G^{\wedge}$ is the union of the groups $\mathfrak{S}_\beta$ defined in 6.1.2, and the prime ideal $\mathfrak{K}_\alpha = \bigcup \{\mathfrak{S}_\beta : \varepsilon_\alpha \varepsilon_\beta \neq \varepsilon_\alpha\}$ plays the role of J_e in Lemma 5.11. Let ξ_α be the characteristic function of $G^{\wedge} \setminus \mathfrak{K}_\alpha$; by 5.11 every generating idempotent of $G^{\wedge\wedge}$ has this form. Now $\xi_\alpha(\chi) = 1$ if and only if $\chi \in \mathfrak{S}_\beta$ with $\varepsilon_\beta \varepsilon_\alpha = \varepsilon_\alpha$, that is, if and only if $N(\chi) = J_\beta \subset J_\alpha$. Thus $\xi_\alpha(\chi) = 1$ if and only if $\tilde{e}_\alpha(\chi) = \chi(e_\alpha) = 1$, by 6.5; hence $\xi_\alpha = \tilde{e}_\alpha$.

6.7 Lemma. *Let $e_\beta \in E$ and let E_β denote the set of all generating idempotents e_α such that $e_\alpha \leqq e_\beta$. Then $e_\beta \in E_\beta^-$.*

Proof. The least idempotent of G belongs to E_β, so E_β is nonvoid. Let U be any neighborhood of e_β; by 6.3 we may assume that

$$U = \{x \in G : |\chi_i(x) - \chi_i(e_\beta)| < \varepsilon, \quad i = 1, \cdots, n\}$$

for some $\varepsilon > 0$ and $\chi_1, \cdots, \chi_n \in G^{\wedge}$. We may also assume that $\chi_i(e_\beta) = 1$ for $1 \leqq i \leqq k$ and $\chi_i(e_\beta) = 0$ for $k < i \leqq n$.

Let $\chi = \chi_1 \cdots \chi_k$ and let $J = N(\chi)$; then J is an open and closed prime ideal, so by 2.7 we have $J = J_\alpha$ for some $\alpha \in A$. Since $e_\beta \in G \setminus J_\alpha$ we have $e_\alpha \leqq e_\beta$, that is, $e_\alpha \in E_\beta$.

For $1 \leqq i \leqq k$, $\chi(e_\alpha) = 1$ implies $\chi_i(e_\alpha) = 1 = \chi_i(e_\beta)$; and for $k < i \leqq n$ we have $\chi_i(e_\alpha) = \chi_i(e_\alpha e_\beta) = \chi_i(e_\alpha)\chi_i(e_\beta) = 0 = \chi_i(e_\beta)$. Thus $e_\alpha \in U \cap E_\beta$, and $e_\beta \in E_\beta^-$.

6.8 Lemma. *Every idempotent of $G^{\wedge\wedge}$ has the form $\tilde{e}_\beta$ for some $e_\beta \in E$.*

Proof. The compact Hausdorff semigroup $G^{\wedge\wedge}$ is also an inverse semigroup with identity, whose semicharacters separate points; thus, Lemma 6.7 applies to $G^{\wedge\wedge}$ as well as to G. If ξ is an idempotent of $G^{\wedge\wedge}$ then either ξ is a generating idempotent or is a limit point of the set of generating idempotents less than ξ. By 6.6 every generating idempotent of $G^{\wedge\wedge}$ is in $\tilde{G}$, so that ξ is in the closure of $\tilde{G}$. But $\tilde{G}$ is compact by 6.4, hence closed. Thus ξ is an idempotent of $\tilde{G}$.

6.9 Remark. Having identified the idempotents of $G^{\wedge\wedge}$ as the functions $\tilde{e}_\alpha$, we can now write $G^{\wedge\wedge} = \bigcup_{\alpha\in A} H_\alpha$, where H_α is the maximal group of elements belonging to $\tilde{e}_\alpha$. To complete the proof we show that $H_\alpha = \tilde{G}_\alpha$ for each $\alpha \in A$, where $\tilde{G}_\alpha = \{\tilde{x} \in G^{\wedge\wedge} : x \in G_\alpha\}$.

6.10 Lemma. *For each $\alpha \in A$ we have $H_\alpha = \tilde{G}_\alpha$.*

Proof. Let $\tilde{x} \in \tilde{G}_\alpha$. Since the mapping $x \to \tilde{x}$ is continuous, and since e_α is the unique idempotent in $\{x, x^2, \cdots\}^-$, it follows that $\tilde{e}_\alpha$ is the unique idempotent in $\{\tilde{x}, \tilde{x}^2, \cdots\}^-$. Thus $\tilde{G}_\alpha \subset H_\alpha$.

Now let $\phi \in H_\alpha$, and suppose first that $\tilde{e}_\alpha$ is a generating idempotent of $G^{\wedge\wedge}$. By Theorem 3.3, $N(\phi) = I(\phi) = I(\tilde{e}_\alpha) = \mathfrak{K}_\alpha$, where $\mathfrak{K}_\alpha = \bigcup\{\mathfrak{S}_\beta : \varepsilon_\beta\varepsilon_\alpha \neq \varepsilon_\alpha\}$. Applying Theorem 4.3 to G we have $\mathfrak{S}_\alpha \cong G_\alpha^\wedge$ by the mapping $\chi \to \chi_\alpha = \chi \mid G_\alpha$. If we define $\phi_\alpha(\chi_\alpha) = \phi(\chi)$, then $\phi_\alpha \in G_\alpha^{\wedge\wedge}$. Hence there exists $x \in G_\alpha$ such that $\phi(\chi) = \phi_\alpha(\chi_\alpha) = \chi_\alpha(x) = \chi(x)$ for all $\chi \in \mathfrak{S}_\alpha$; that is, $\phi \mid \mathfrak{S}_\alpha = \tilde{x} \mid \mathfrak{S}_\alpha$. Applying 4.3 to $G^\wedge$ (with $\mathfrak{K}_\alpha$ in place of J_e), we see that $\phi \mid \mathfrak{S}_\alpha$ admits a unique extension in H_α. Thus $\phi = \tilde{x}$, and $H_\alpha \subset \tilde{G}_\alpha$.

Finally, let H_β be arbitrary and let $\phi \in H_\beta$. By 6.7 there is a net of generating idempotents $\tilde{e}_\alpha$ such that $\tilde{e}_\alpha \leqq \tilde{e}_\beta$ and $\tilde{e}_\beta = \lim \tilde{e}_\alpha$. Then $\phi = \phi\tilde{e}_\beta = \lim \phi\tilde{e}_\alpha$. Since $\phi\tilde{e}_\alpha \in H_\alpha \subset \tilde{G}$, we have $\phi \in \tilde{G}$. Since the groups H_β are disjoint and $\tilde{G}_\beta \subset H_\beta$ for each $\beta \in A$ we have in fact $\phi \in \tilde{G}_\beta$.

The above results are summarized in the following theorem:

6.11 Theorem. *Let G be a commutative compact Hausdorff inverse semigroup with identity, such that $G^\wedge$ separates points. Then $G \cong G^{\wedge\wedge}$ under the mapping $x \to \tilde{x}$.*

6.12 Example. If G is a commutative compact Hausdorff inverse semigroup such that every idempotent is a generating idempotent, then $G^\wedge$ separates points (use 4.3 and the fact that $G_\alpha^\wedge$ separates points of G_α). This condition is not necessary, however, as the following example shows. Let S be an infinite set, let K be the multiplicative semigroup of complex numbers z with $z = 0$ or $|z| = 1$, and with its topology as a subspace of the plane. Let $G = K^S$, with the product topology and pointwise multiplication. For each set $A \subset S$ let

$$G_A = \{f \in G : f(s) \neq 0 \text{ if and only if } s \in A\}.$$

Then G_A is a group and is in fact the group of elements belonging to the idempotent e_A, where e_A is the characteristic function of A. The Schwarz decomposition of G is $G = \bigcup\{G_A : A \subset S\}$, so that G is a commutative compact Hausdorff inverse semigroup. Every projection $\pi_s : f \to f(s)$ is a semicharacter of G, so $G^{\wedge}$ separates points. But e_A is a generating idempotent if and only if A is finite. (The operation of Lemma 6.7 is made more clear by this example also).

REFERENCES

1. A. H. Clifford, *Bands of semigroups*, Proc. Amer. Math. Soc. **5** (1954), 499–504.
2. ———, *Semigroups admitting relative inverses*, Ann. of Math. (2) **42** (1941), 1037–1049.
3. E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70–97.
4. ———, *Finite dimensional convolution algebras*, Acta Math. **93** (1955), 67–119.
5. W. D. Munn and R. Penrose, *A note on inverse semigroups*, Proc. Cambridge Philos. Soc. **51** (1955), 396–399.
6. K. Numakura, *On bicompact semigroups*, Math. J. Okayama Univ. **1** (1952), 99–108.
7. G. B. Preston, *Inverse semigroups*, J. London Math. Soc. **29** (1954), 396–403.
8. Š. Schwarz, *On the theory of Hausdorff bicompact semigroups*, Czechoslovak Math. J. **5(80)** (1955), 1–23. (Russian).
9. ———, *Characters of bicompact semigroups*, Czechoslovak Math. J. **5(80)** (1955), 24–28.
10. ———, *The theory of characters of commutative Hausdorff bicompact semigroups*, Czechoslovak Math. J. **6(81)** (1956), 330–361.
11. R. J. Warne and L. K. Williams, *Characters on inverse semigroups*, Czechoslovak Math. J. **11(86)** (1961), 150–155.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASHINGTON
UNIVERSITY OF COLORADO,
BOULDER, COLORADO